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E. S. HOLDEN, J. E. KEELER, C. G. YALE,
Committee on Publication.

ON THE CRITERION OF CONTINUITY OF FUNCTIONS OF A REAL VARIABLE AND ON THE THEOREM OF MEAN VALUE.

By IRVING STRINGHAM, PH. D.

The following geometrical representation presents the criterion of continuity of a function of a real variable in an interesting form, and leads to a sort of generalized theorem of mean value expressing a relation between the *incremental* and *decremental* derivatives.

THE GEOMETRICAL REPRESENTATION.

Let X be a point of the curve represented by the equation $y = f(x)$. (Figure 1.) Draw two parallel chords, PQ and $P'Q'$, the latter intercepting the curve at two points, P', Q', near X on either side, and suppose x to be a continuously increasing or decreasing variable within the interval PQ, assuming once only all real values between a and b, where

$$OM = a,$$
$$ON = b = a + h,$$

so that the abscissa of

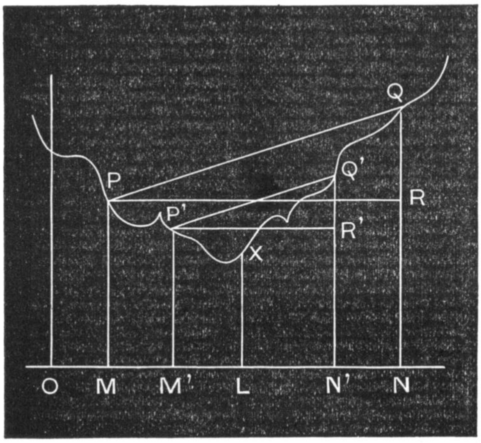

FIGURE 1.

any point between P and Q will lie, in numerical value, between a and b. Complete the figure by drawing through P, P', Q, Q', etc., lines parallel to the co-ordinate axes, and let

$$MN = h, \quad ML = \theta h,$$
$$M'N' = k, \quad M'L = \epsilon k,$$

so that the values of θ and ϵ lie between 0 and + 1. Then

$$MM' = \theta h - \epsilon k, \quad OM' = a + \theta h - \epsilon k,$$
$$OL = a + \theta h, \quad ON' = a + \theta h - \epsilon k + k;$$

and by similar triangles

$$\frac{RQ}{PR} = \frac{R'Q'}{P'R'},$$

that is

$$\frac{f(a+h) - f(a)}{h} = \frac{f(a + \theta h - \epsilon k + k) - f(a + \theta h - \epsilon k)}{k},$$

with the conditions

$$1 \geqq \theta \geqq 0, \quad 1 \geqq \epsilon \geqq 0.$$

This will be referred to as the *difference equation.*

THE CRITERION OF CONTINUITY.

If the difference equation persist while ϵ varies continuously between 0 and + 1, LX, or $f(a + \theta h)$, will be a true ordinate to the curve for every value of ϵ so chosen, and the function will be continuous throughout the interval $P'Q'$; it will be discontinuous for the values of ϵ that make the difference equation impossible. Hence, the ordinate $f(a + \theta h)$ will vary continuously within the entire interval PQ, provided, independently of the value of ϵ and for every value of h between zero and MN, the ordinates $M'P'$, $N'Q'$ have a common limiting value as they approach each other,* or, what is the same thing, provided the limiting value of their difference is zero when k approaches zero. Hence the criterion of continuity for the function within the interval PQ may be stated as follows:

If for every value of ϵ *between* 0 *and* + 1 *and every value of* h *between* 0 *and* b — a, *the condition*

$$\underset{k = 0}{\text{limit}} \left\{ f(a + \theta h - \epsilon k + k) - f(a + \theta h - \epsilon k) \right\} = 0,$$
$$h \geqq k \geqq 0, \quad 1 \geqq \theta \geqq 0,$$

be satisfied, then the function f (x) *is continuous throughout the interval from* a *to* b.

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* Compare Harnack, *Differential u. Integral Rechnung*, p. 25.

In particular, the function is continuous within the indefinitely small interval $P'Q'$, defined in position by the abscissa $[a + \theta h]_{k=0}$, *if for every value of* ϵ *between* 0 *and* +1

$$\operatorname*{limit}_{k=0} \left\{ f\,(a + \theta h - \epsilon k + k) - f\,(a + \theta h - \epsilon k) \right\} = 0,$$

$$1 \geqq \theta \geqq 0.$$

Any value of $[a + \theta h]_{k=0}$ *that permits the failure of this condition defines a point of discontinuity.*

In applying the criterion, the abscissa $a + \theta h$ will, of course, usually be an assumed and therefore a known quantity.

The Theorem of Mean Value.

In order to investigate the limiting value of the difference ratio for $k = 0$ in critical cases, it is expedient to write the difference equation in the form

$$\frac{f\,(a + h) - f\,(a)}{h} = (1 - \epsilon)\frac{f\,(a + \theta h - \epsilon k + k) - f\,(a + \theta h)}{-\epsilon k + k}$$
$$+ \epsilon\frac{f\,(a + \theta h - \epsilon k) - f\,(a + \theta h)}{-\epsilon k}.$$

This is obtained by adding $f\,(a + \theta h)$ to the term $-f\,(a + \theta h - \epsilon k)$ and subtracting it from $f\,(a + \theta h - \epsilon k + k)$. The coefficients of $(1 - \epsilon)$ and ϵ in this equation I call the *incremental* and *decremental* difference-ratios respectively, and the corresponding derivatives, obtained by passing to the limit for $k = 0$, I name the *incremental* and *decremental* derivatives;* for brevity, these will be referred to collectively as the two critical derivatives. In order to distinguish the derivatives from each other symbolically, the former will be denoted by $f'_{+}\,(a + \theta h)$, the latter by $f'_{-}\,(a + \theta h)$.

If in the difference equation in its revised form we are allowed to pass to the limit for $k = 0$, it becomes (since ϵ does not in general vanish with k),

$$\frac{f\,(a + h) - f\,(a)}{h} = (1 - \epsilon)\,f'_{+}\,(a + \theta h) + \epsilon\,f'_{-}\,(a + \theta h).$$

Now, so long as the function remains continuous and one-valued, k may be made, by virtue of the criterion of continuity, to pass to its limit zero, and if then the critical derivatives are determinate, the above equation exists for some value of ϵ and some value of θ, satisfying the conditions $1 \geqq \epsilon \geqq 0,\ 1 \geqq \theta \geqq 0$. This result may be stated in the following formal theorem:

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* In German "Vor- und Rückwärts-genommene Differential-quotienten." See Harnack *Differential u. Integral Rechnung*, p. 33.

If, within the interval from a *to* a + h, f (x) *be a continuous and one-valued function, and if its incremental and decremental derivatives are, throughout this interval, separately finite and determinate, then for some value of* ε *and for some value of* θ *satisfying the conditions*

$$1 \geqq \epsilon \geqq 0,\quad 1 \geqq \theta \geqq 0,$$

there exists a value a + θh, *of the variable, intermediate between* a *and* a + h, *such that*

$$\frac{f(a+h)-f(a)}{h} = (1-\epsilon)\, f'_+(a+\theta h) + \epsilon\, f'_-(a+\theta h).$$

This theorem obtains for all finite determinate values of the critical derivatives, however great they may be; and since $\frac{f(a+h)-f(a)}{h}$ is finite by construction, if $f'_+(a+\theta h)$ approaches ± ∞, either ε becomes unity or $f'_-(a+\theta h)$ approaches $\mp \infty$; but if $f'_-(a+\theta h)$ approaches ± ∞ while $f'_+(a+\theta h)$ remains finite, then ε beeomes zero. Hence the following corollary:

If at a point at which the function is continuous and one-valued, one of the critical derivatives becomes $+\infty$, *either the other becomes* $-\infty$ *or else* ε *is zero or unity.*

Hence, also, if the critical derivatives are equal, they are finite (may be zero); and again, if they are equal, the function being continuous and one-valued, the corresponding critical difference-ratios approach a definite limit when k approaches zero. The above theorem of mean value then takes the special form

$$\frac{f(a+h)-f(a)}{h} = f'(a+\theta h),$$

which is Rolle's Theorem.

In particular, if $f'_+(a+\theta h)$ and $f'_-(a+\theta h)$ are both zero for all values of h within the interval considered, then $f(a+h)$ remains throughout equal to $f(a)$, a constant.

These results may be briefly recapitulated as follows: The function being continuous and one-valued within the interval considered, the values of the critical derivatives and the forms of the theorem of mean value correspond in the following manner, where $f'(a, h)$ stands for $f\frac{(a+h)-f(a)}{h}$.

If the critical derivatives are finite and different,

$$f'(a, h) = (1-\epsilon) f'_+(a + \theta h) + \epsilon f'_-(a + \theta h).$$

If both are infinite,

$$f'(a, h) = \pm \infty \mp \infty .$$

If only one is infinite,

$$\text{either } f'(a, h) = f'_+(a + \theta h) \pm \underset{k=0}{\text{limit}} \left\{ \epsilon f'(a + \theta h, -\epsilon k) \right\},$$

$$\text{or } f'(a, h) = f'_-(a + \theta h) \pm \underset{k=0}{\text{limit}} \left\{ (1-\epsilon) f'(a + \theta h, -\epsilon k + k) \right\}.$$

If identical (cannot be infinite),

$$f'(a, h) = f'(a + \theta h). \quad \text{[Rolle's Theorem.]}$$

If both are zero,

$$f'(a, h) = 0, \quad f(x) = \text{constant.}$$

If only one is zero,

$$f'(a, h) = \epsilon f'_-(a + \theta h), \text{ or } = (1-\epsilon) f'_+(a + \theta h).$$

THE CIRCLE OF CURVATURE.

Through the three points P', Q', X of Figure 1, let a circle be passed, as shown in Figure 2, and suppose the chords $P'X$, $Q'X$ to intersect the x-axis in E and F, respectively.

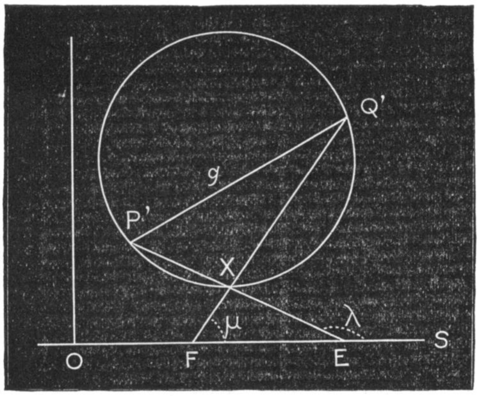

FIGURE 2.

Let

$P'Q' = g$,
angle $P'ES = \lambda$,
angle $Q'FS = \mu$,
R = radius of circle.

Then

$$R = \frac{g}{2 \sin (\lambda - \mu)}.$$

If $c = a + \theta h$, the co-ordinates of P' and Q' are:

of P', $c - \epsilon k$, $\quad f(c - \epsilon k)$,
of Q', $c - \epsilon k + k$, $f(c - \epsilon k + k)$.

Hence,

$$g^2 = k^2 + [f(c - \epsilon k + k) - f(c - \epsilon k)]^2$$

$$= k^2 + k^2 \left\{ (1 - \epsilon) \frac{f(c - \epsilon k + k) - f(c)}{-\epsilon k + k} + \epsilon \frac{f(c - \epsilon k) - f(c)}{-\epsilon k} \right\}^2$$

$$= k^2 + k^2 [(1 - \epsilon) \tan \mu + \epsilon \tan \lambda]^2;$$

$$\therefore R = \pm \frac{k}{2} \frac{\sqrt{1 + [\epsilon \tan \lambda + (1 - \epsilon) \tan \mu]^2}}{\sin(\lambda - \mu)},$$

$$= \pm \frac{k}{2} \frac{\sec \lambda \sec \mu \sqrt{1 + [\epsilon \tan \lambda + (1 - \epsilon) \tan \mu]^2}}{\tan \lambda - \tan \mu}.$$

Evidently the condition for the existence of a circle of curvature is

$$\lambda - \mu = \pm 180^\circ, \text{ when } k = 0.$$

When this condition is fulfilled, the above expression for *R* easily reduces to the well-known expression for radius of curvature.

BERKELEY, July, 1889.

DOES THE COLOR OF A STAR INDICATE ITS AGE?

By WILLIAM M. PIERSON.

Much discussion has been indulged in as to the relative ages of the stars, and widely divergent conclusions have resulted. ZOELLNER, in 1865, maintained that yellow and red stars are simply white stars in various stages of cooling. D'ARREST refused to adopt the theory. VOGEL, however, in 1874, classified the spectra of the stars on ZOELLNER's theory as being the "rational" basis. Father SECCHI leaned towards the same theory. Professor YOUNG indicates great doubts as to its correctness, and, in that connection, says: "But it is very far from certain that a red star is not just as likely to be younger than a white one, as to be older. It probably is now at a lower temperature, and possesses a more extensive envelope of gases, but it may be increasing in temperature as well as decreasing. At any rate, we have no certain knowledge about its age." (*Gen'l Astr.*, Sec. 858.) M. JANSSEN, Director of the Observatory at Meudon, France, has, however, maintained the same theory. An abstract of his valuable paper, read before the Institute of France, will be found in Volume 7 of the *Sidereal Messenger* (p. 202). His conclusions are that in-